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Processing Decimal Values

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Processing Decimal Data

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Objective: Correct Decimal Processing

- Round-Trip: Decimal input comes back as the exact same value on output.
- Correct Basic Maths:
  - In case of too many digits, at least they are correctly rounded.
  - Otherwise, addition, subtraction, and multiplication are exact.
  - Division, fractional power, etc. generally can't be exact.

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Number of Digits

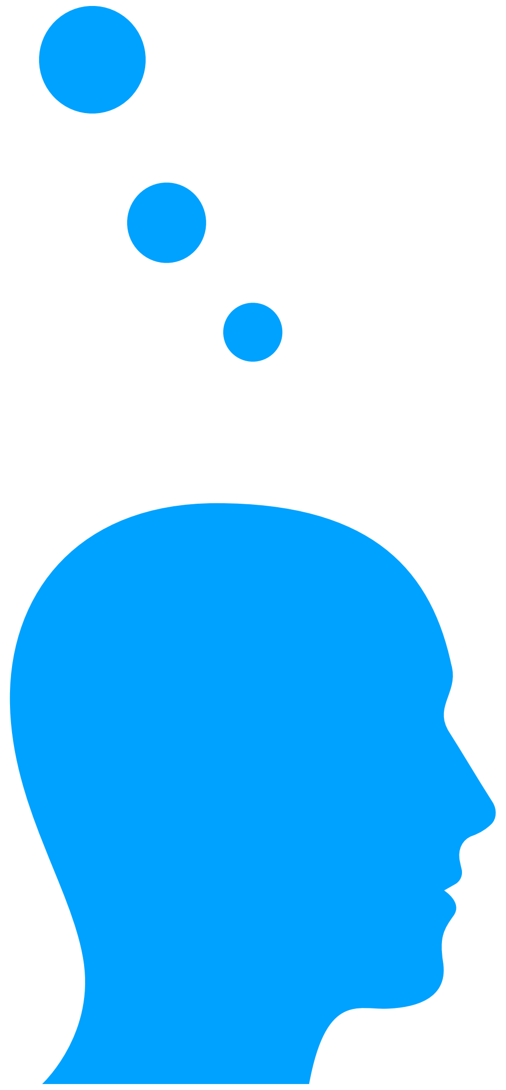
- The number of digits grows…
  - … for addition to use an extra digit per addition.
  - … for multiplication to use the sum of the factors’ digits.
- The number of digits in fixed size representations is limited.
- For exact computations, the number of digits needs to be controlled.

The Problem

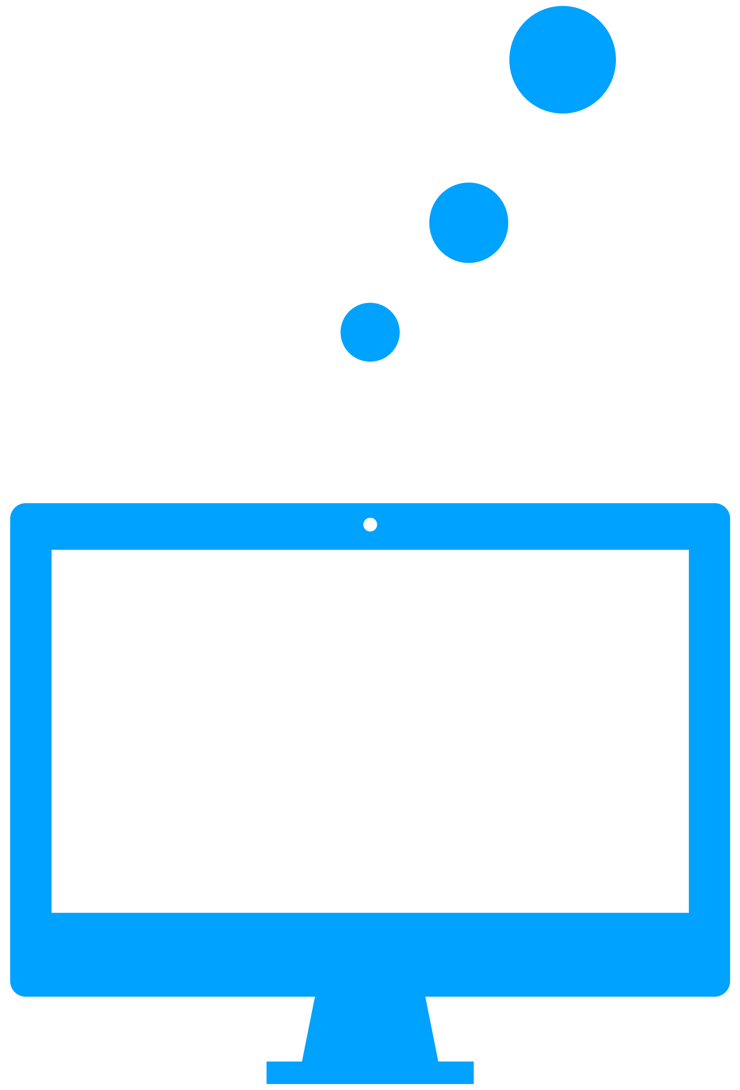
0.1

The Problem (double)

0.1



0.1000000000000000055511151231257827021181583404541015625



0.1

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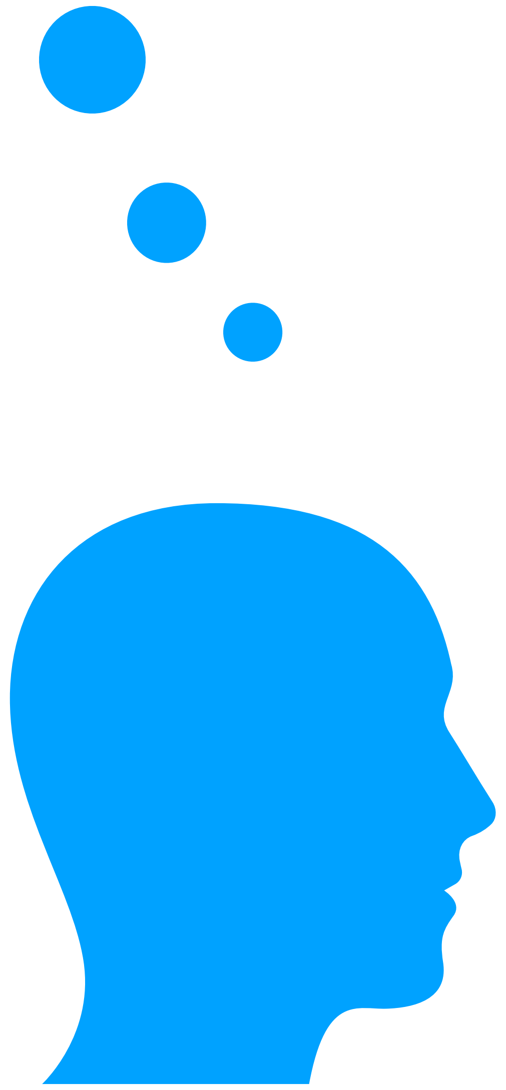
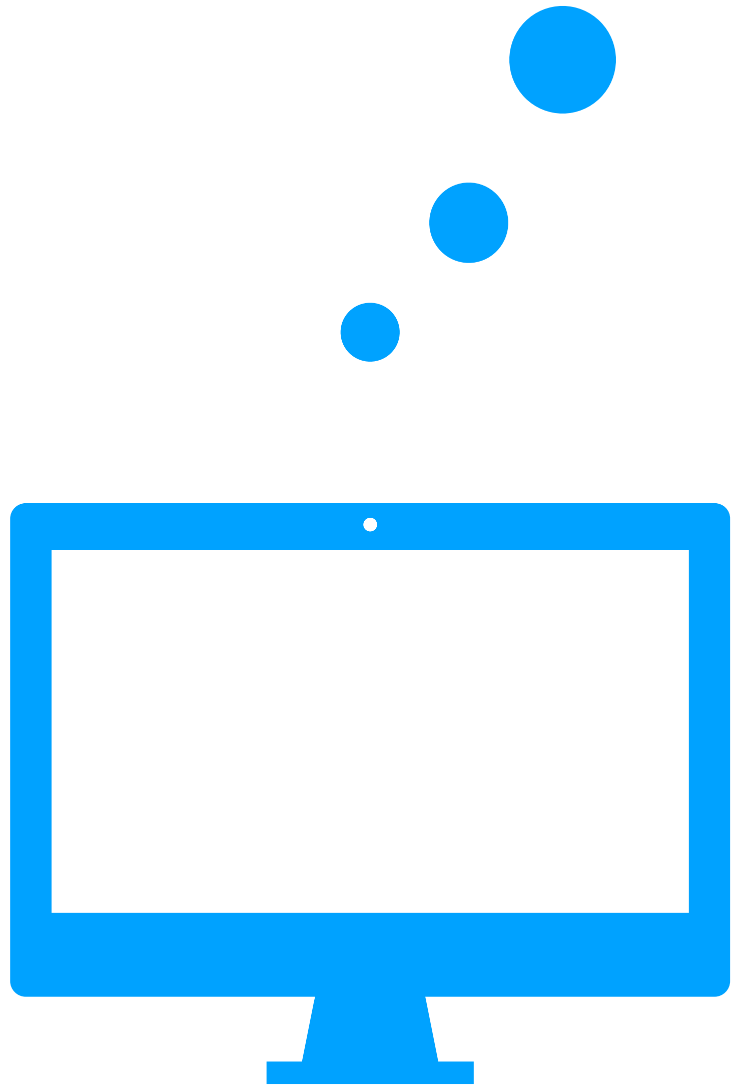
The Problem (float)

0.1



0.1

0.0999999940395355224609375



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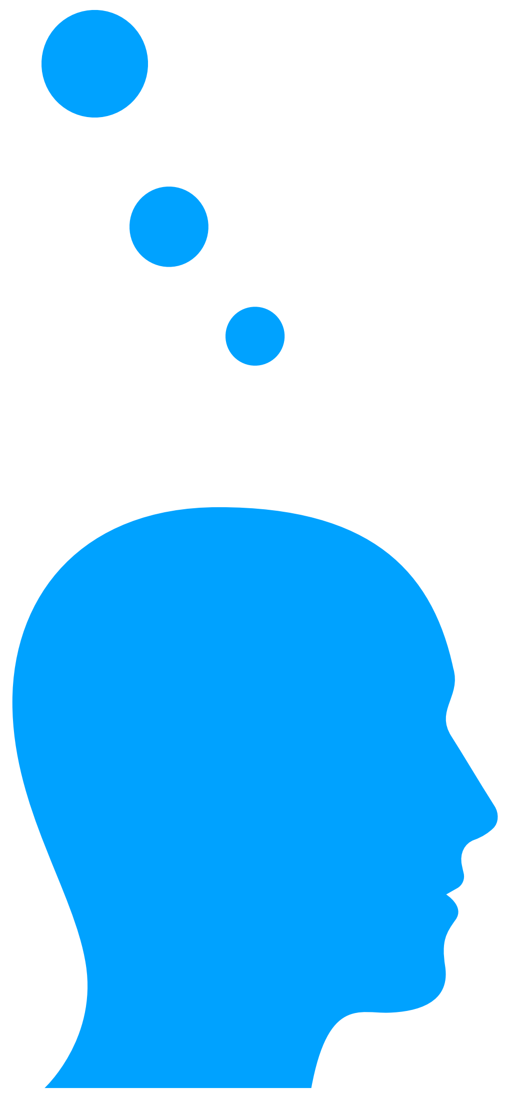
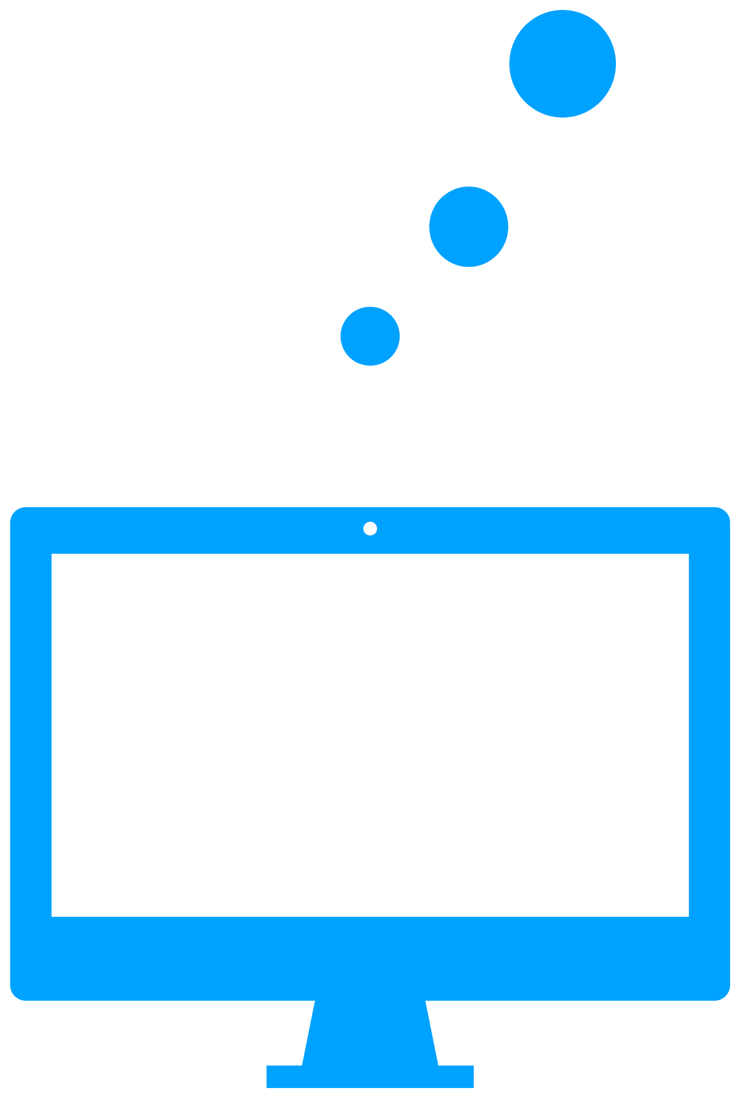
The Problem (float)

0.1



1.10011001100110011001100b-4

0.1



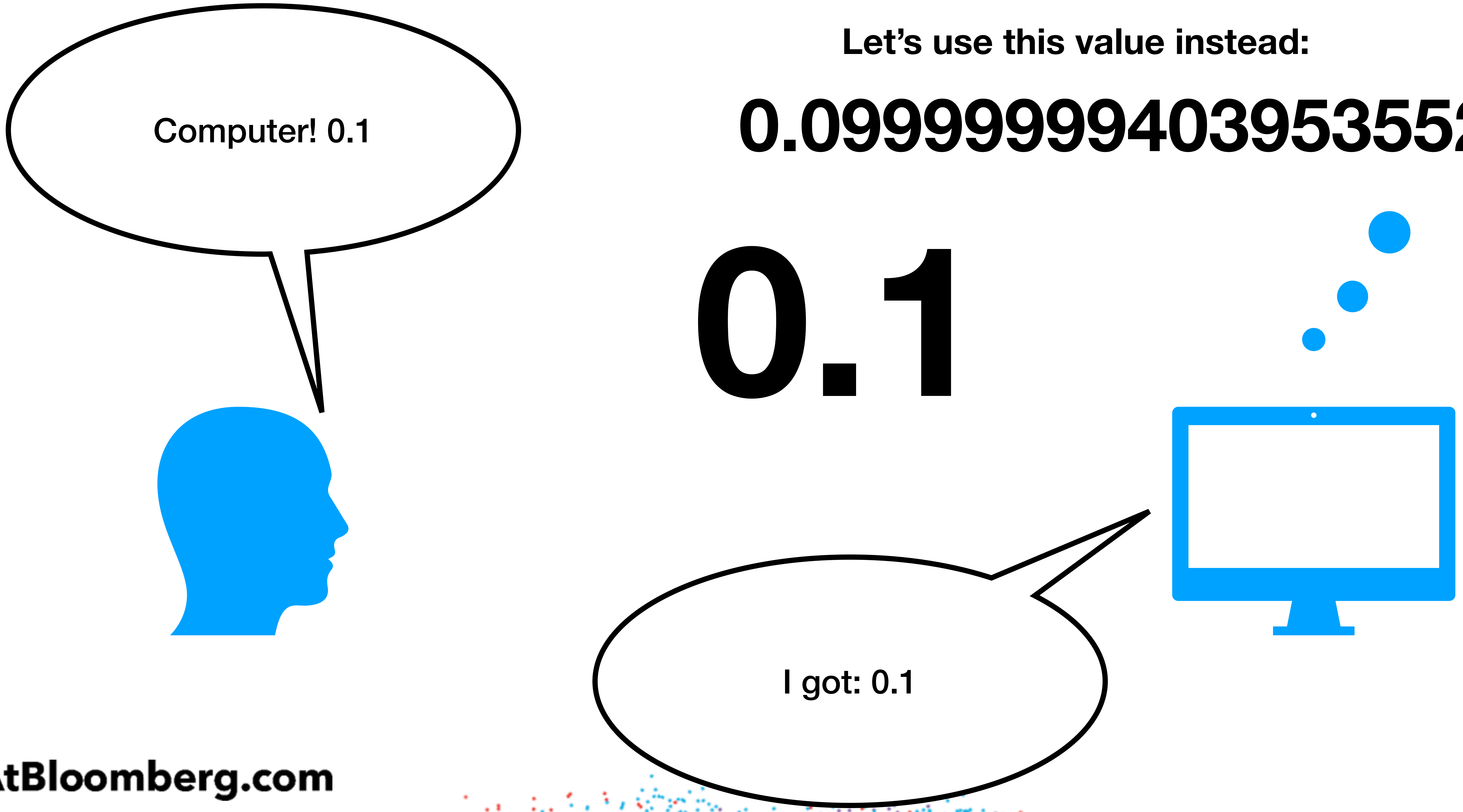
The Problem

- Integer values do not have a problem.
- For fractional values, a floating point representation is used:
  - Due to available hardware, *binary* floating points are used.
  - A binary representation cannot represent all decimal values exactly.
  - The problem is masked by looking as if things work.

The Problem (float)

Let’s use this value instead:

0.0999999940395355224609375



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Expectations

- Basic arithmetic operations work correctly.

- Nothing really esoteric, just some simple expressions:

```
0.3 + 0.6 == 0.9
0.4 - 0.3  == 0.1
0.3 * 3     == 0.9
```

- Sadly: none of the above is true for float or double.

The Representation

- Values get decomposed into three components:
    - The values of the components depend on the used *base*.
    - The *sign* of the value: + or -
    - An integer used as *exponent* for the base to scale the value.
    - An integer to represent the unscaled value (called *significand*).

The Representation

$$\pm 0\ldots 0 d_i d_{i-1} \ldots d_0 . d_{-1} \ldots d_{-j} 0 \ldots 0$$

$d_i \in [0, \text{base})$

The Representation

$$(-1)^{sign} * \mathbf{base}^{exponent} * \sum_{i \in [0,\infty)} d_i * \mathbf{base}^{i}$$

The Representation

$$(-1)^{sign} * \text{base}^{exponent} * \sum_{i \in [0,\#digits)} d_i * \text{base}^i$$

The Representation

$$(-1)^{\mathit{sign}} * \mathbf{base}^{\mathit{exponent}} * \mathit{significand}$$

The Representation: Special Case Integer

$$(-1)^{sign} * significand$$

Computers don't really use that: using two's complement makes things a bit simpler.

The Representation: Special Case Unsigned Integer

significand

Encoding Decimal Values

- Use the closest representable values.
- Minimizes errors on computations.
- Allows round-trip of decimal values (subject to reasonable constraints):
  - The decimal value can be restored from the binary representation.
  - Assuming not too many digits are used and the value is in range.
  - Trailing zeros can’t be recovered.

Encoding Decimal Values: 0.1

Digit Value:

- 0 * 0.5
- 0 * 0.25
- 0 * 0.125
- 1 * 0.0625
- 1 * 0.03125
- 0 * 0.015625
- 0 * 0.0078125
- 1 * 0.00390625

Remaining value:

- 0.1
- 0.1
- 0.1
- 0.0375
- 0.00625
- 0.00625
- 0.00625
- 0.00234375

Encoding Decimal Values

- Two related papers:
  - “How to Read Floating Point Numbers Accurately”, Clinger, https://dl.acm.org/doi/pdf/10.1145/93542.93557
  - “How to Print Floating-Point Numbers Accurately”, Steele/White, https://dl.acm.org/doi/pdf/10.1145/93548.93559
    In particular Dragon 4 for general printing.
- Better performance algorithms for Printing: Grisu and Ryu.

Dragon Algorithms Idea (Recovering Decimal Value)

- Determine the decimal value closest to the encoded binary value.
- To do so, produce leading digits and track the size of the remaining error:
  - Once the error becomes bigger than the remaining value, stop!
- Implication: the binary value correctly represents a decimal value.

Round-Trip

- Represent decimal value as binary FP and restore decimal value
  - Assumes the decimal is in the range the binary value can cover
  - There are a limited number of decimal digits:
    - float: 6, double: 15
  - Trailing, fractional zeros are lost (numeric value is the same, though)

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Why Can Float Only Round-Trip 6 Digits?

- Float uses 24 bits for the significand:
  - 10 bits can represent 1024 values, 3 decimal digits.
  - 4 bits can easily represent one decimal digit.
- Problem: the values are not evenly distributed.
- Example problem:
  - Identical representation for 9.536745e-07 and 9.536746e-07 (0x35800002)

Base 10: Exact Representation of Decimal Value

- Subtraction, addition, multiplication can produce exact values.
- Comparison and formatting readily produce correct results.
- Decimal rounding can be done correctly.

Decimal Representations: String

- The usual representations when processing text.
- BCD (Binary Coded Decimal) packs the data more tightly:
  - 4 bits per digit (or sign or, possibly, decimal point).
- Problems:
  - Variable size or a relatively small range of value.
  - Computations are relatively slow.

Decimal Fixed Point: Scale by a Fixed Power of 10

- Representation is just a signed integer: decimal point implicit in the type.
- Advantage: Operations are very fast - just integer operations.
- Disadvantage: the scale needs to be known and fixed.
- FixedPoint<N> + FixedPoint<N> => FixedPoint<N>
- FixedPoint<N> * FixedPoint<M> => FixedPoint<N + M>

Decimal FixedPoint: Different Scaling Factors

- FixedPoint<N> + FixedPoint<M> => FixedPoint<max(N, M)>
  - Not quite as fast: requires a multiplication by $10^{abs(N - M)}$.
- Often a suitable, constant scaling factor isn't known.
- Make it more flexible: don't encode the scaling in the type!
  - Use scaling factor from context: becomes more fragile.
  - Idea: store the scaling factor together with the value!

Decimal Floating Point: Scale by Variable Power of 10

- Representation is similar to binary floating point.
- The representation is not normalized:
  - Equal values may have multiple representations: *cohorts*.
  - Allows representation of number of trailing zeros.
  - Although equal these may display differently, e.g., 0.1 and 0.10.
- Standardized by IEEE 754 (2008)

Decimal Floating Point

- IEEE 754 DFP use ~54 bits for the signficand, ~9 bits for the exponent, and 1 bit sign.
- Scaling for addition may require division by a power of 10:
    - Fixed set of divisors needed: use precomputed values with multiplication.
    - Idea: instead of division by 10^n, multiply by $2^k * 10^{-n}$.
- Typical use cases often sum values with the same scale.
- The flexibility has some cost.

Multiplication Instead of Division

- a / b
- $== a * 1 / b$
- $== a * 2^n / (b * 2^n)$
- $== a * (2^n / b) / 2^n$
- $\approx a * \lfloor 2^n / b \rfloor / 2^n$

- Choose n such that the error doesn't matter.

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Decimal Floating Point: bdldfp Implementation

- With C++, DFPs can be represented as a suitable class.
- There is an open source implementation as part of BDE.
- bdldfp::Decimal64 (and bdldfp::Decimal32).
- Implemented using Intel's open source C implementation.
- https://github.com/bloomberg/bde/tree/master/groups/bdl/bdldfp

Conversions From Binary To Decimal Floating-Point

- Value preserving: fast, but doesn't restore encoded decimal values.

- Decimal value restoring: slow, but get back the original value.

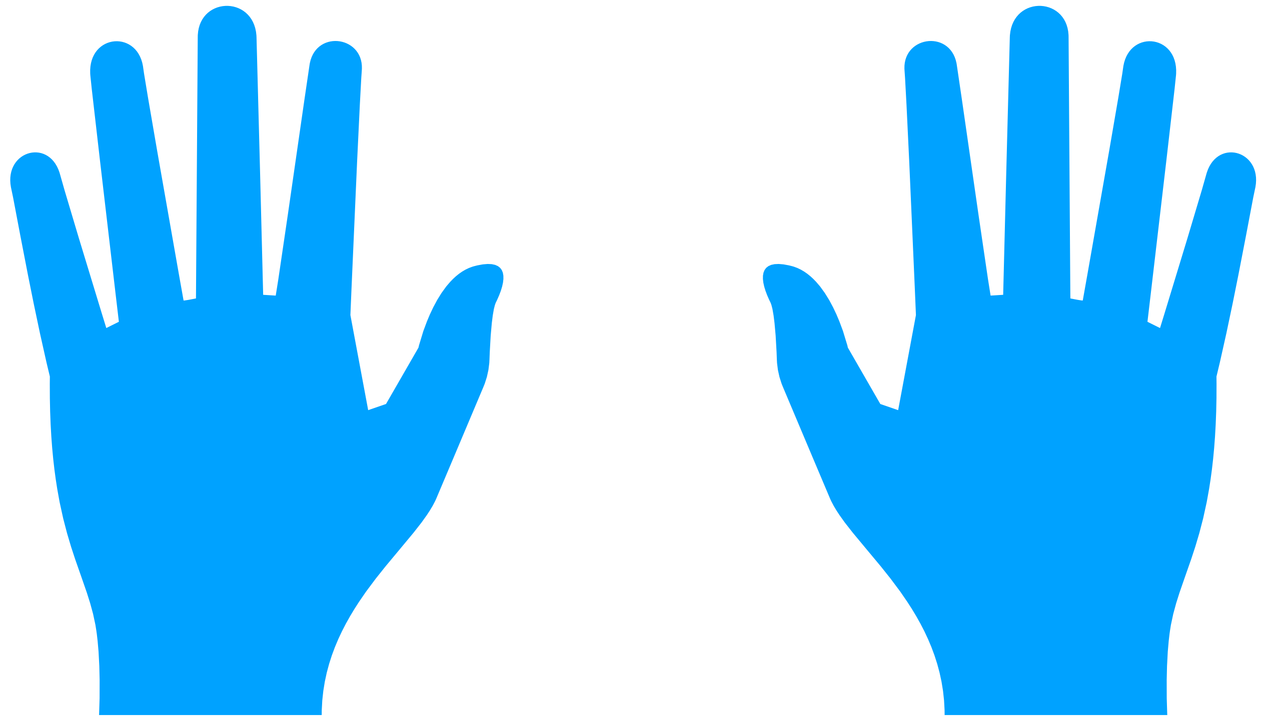
- Which one to use depends on the context.

Decimal vs. Binary: What to Use

- For exact computation, e.g., in finance: decimal.
  - The transport can be binary if necessary, e.g., for Excel plug-ins.
- Any estimate, simulation, etc.: binary.
  - Exposing a decimal does allow control over the result.
- The value of fractional powers, e.g. interest rates, can't be represented exactly.

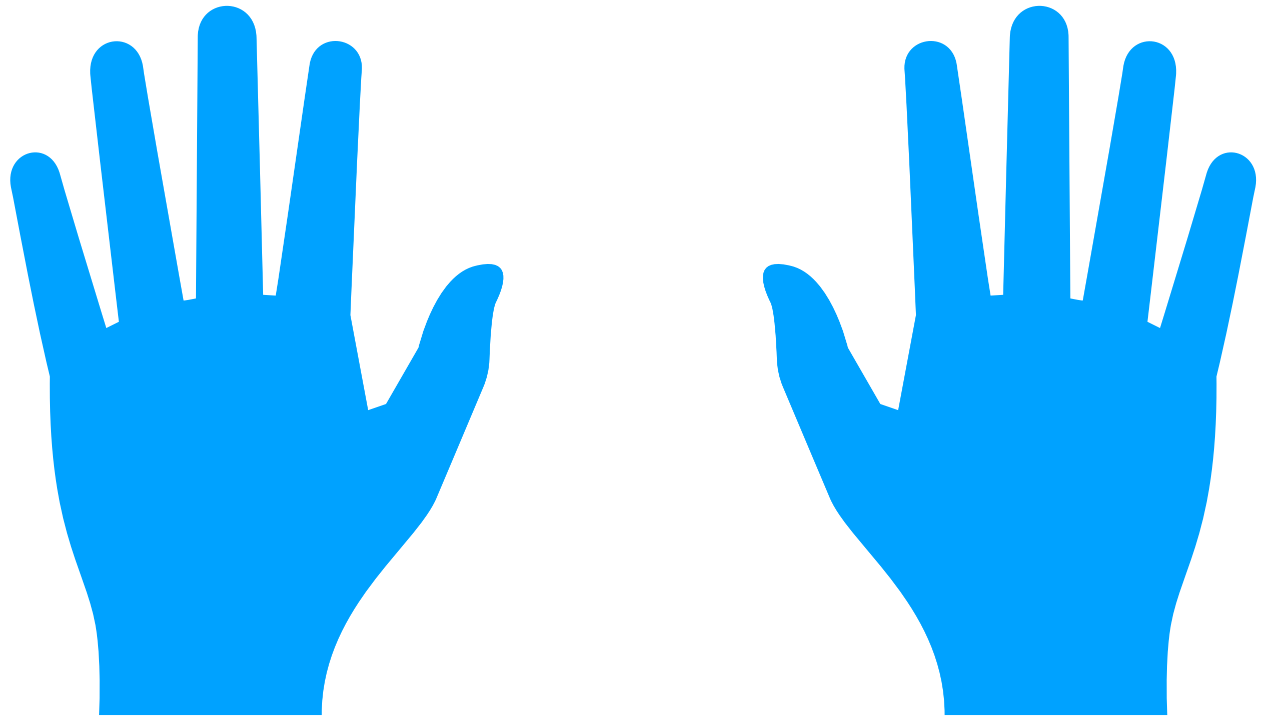
Nature Primed Us Well

- We have 8 fingers: we should use these to count!



Nature Primed Us Well

- We have 8 fingers: we should use these to count!
- Sadly, someone had the bad idea to also use the thumbs…



Thank you!

Questions?

References

- BDE: https://github.com/bloomberg/bde/tree/master/groups/bdl/bdldfp
- IEEE 754 analyzer: https://babbage.cs.qc.cuny.edu/IEEE-754/
- Printing Floating Points: https://dl.acm.org/doi/pdf/10.1145/93548.93559
- Reading Floating Points: https://dl.acm.org/doi/pdf/10.1145/93542.93557